The set of all real numbers, the set of all complex numbers, and the set of all quaternions are all examples of something called an associative algebra. The associative part just means multiplication is associative – $(ab)c = a(bc)$ – the algebra part means they are a set of mathematical objects that are defined by their basis. For the real numbers the basis is the set {1}, for the complex numbers the basis is the set {1,i}, and for the quaternions the basis is the set {1,i,j,k}. Every element of an algebra can be written as a polynomial in the first degree where each term is equal to a real number times some basis unit. That is to say, the real numbers can all be represented in the form a, the complex numbers in the form $a + bi$, and the quaternions in the form $a + bi + cj + dk$, or more commonly, some quaternion q is written as $q_1 + q_2i + q_3j + q_4k$. The defining formula for the quaternions is:

$$i^2=j^2=k^2=ijk=-1$$

From this formula we can derive the following:

$$ij=k$$
$$ji=-k$$
$$jk=i$$
$$kj=-i$$
$$ki=j$$
$$ik=-j$$

This demonstrates that the basis unit of the quaternions do not commute, that is to say ab is not neccesarily equal to ba. In fact, it is evident that they anticommute: $ab = -ba$. As such, the basis units are anticommutative, but quaternions in general may or may not commute; they can also neither commute nor anticommute, in fact, this is usually the case. We will go into more depth on the subject of multiplication in the multiplication lesson.

The complex plane is a wonderful way to visualize the idea of a two dimensional number, unfortunately, there is no convenient quaternionic equivalent, because we can't really illustrate 4-dimensional space.

For some quaternion $q_1 + q_2i + q_3j + q_4k$, we define it as having a scalar part and a vector part, with the scalar part being q_1 and the vector part being $q_2i + q_3j + q_4k$. Generally, these are written as a and v, respectively. Quaternions also have a defined conjugate $a - v$, or $q_1 – q_2i – q_3j – q_4k$.

Similar to the case with complex numbers, quaternionic addition and subtraction is defined partwise, there are just more parts.

$$(q_1+q_2i+q_3j+q_4k)+(p_1+p_2i+p_3j+p_4k)=(q_1+p_1)+(q_2+p_2)i+(q_3+p_3)j+(q_4+p_4)k$$
$$(q_1+q_2i+q_3j+q_4k)-(p_1+p_2i+p_3j+p_4k)=(q_1-p_1)+(q_2-p_2)i+(q_3-p_3)j+(q_4-p_4)k$$

That about sums up the first lesson on quaternions.

Optional Exercises:

1. (1 + 3i – 4j + 2k) + (-2 + 2i + j – 7k)

2. (1 + 4i – 6k) + (3i – 7j – 10k)

3. (1 + 2i + 5j – 6k) – (1 + 3i – 2j – 7k)